
UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical & Computer Engineering
ECE 204 Numerical methods
The adaptive Euler-Heun method
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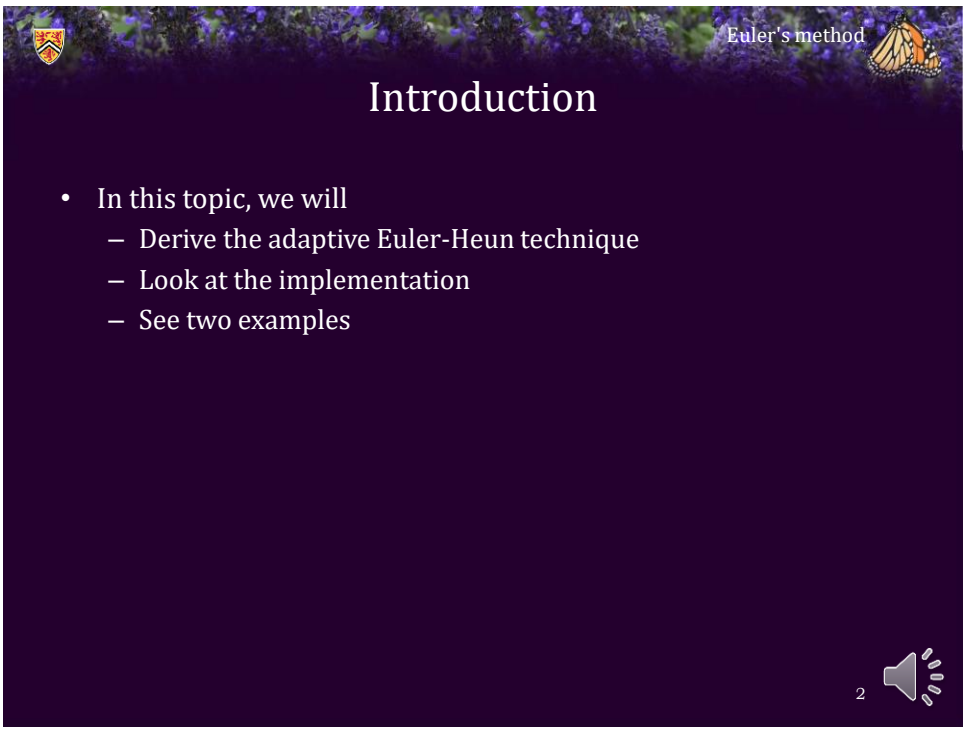
Euler's method
Introduction
• In this topic, we will
– Derive the adaptive Euler-Heun technique
– Look at the implementation
– See two examples
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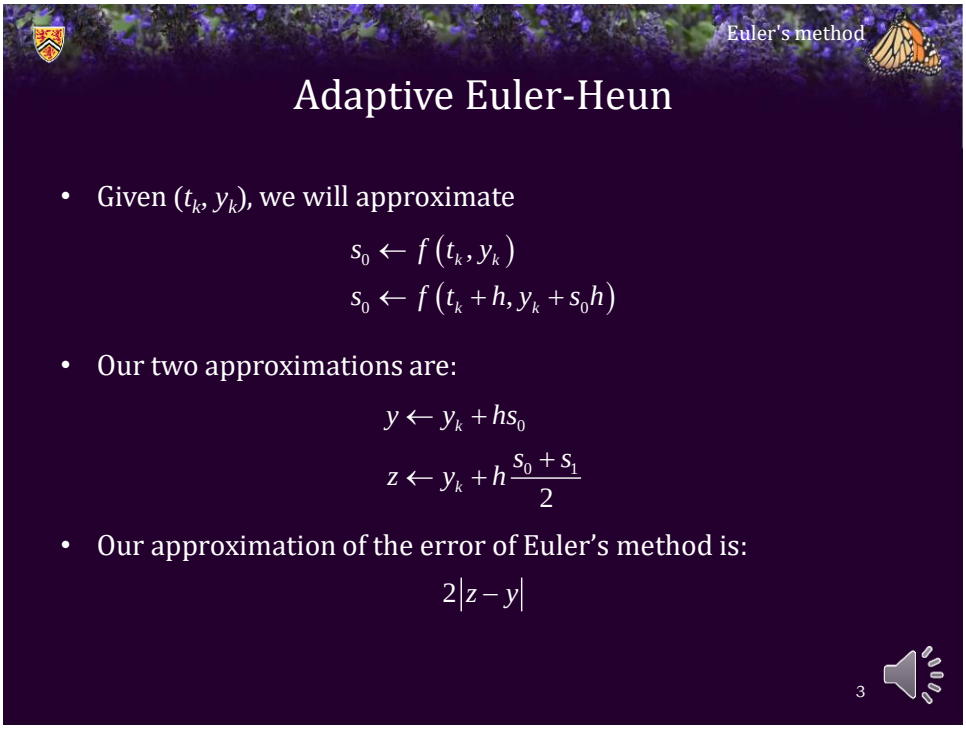
Euler's method
Adaptive Euler-Heun
• Given (t_k, y_k), we will approximate
$s_0 \leftarrow f(t_k, y_k)$
$s_0 \leftarrow f(t_k + h, y_k + s_0 h)$
• Our two approximations are:
$y \leftarrow y_k + h s_0$
$z \leftarrow y_k + h \frac{s_0 + s_1}{2}$
• Our approximation of the error of Euler's method is:
$2|z - y|$
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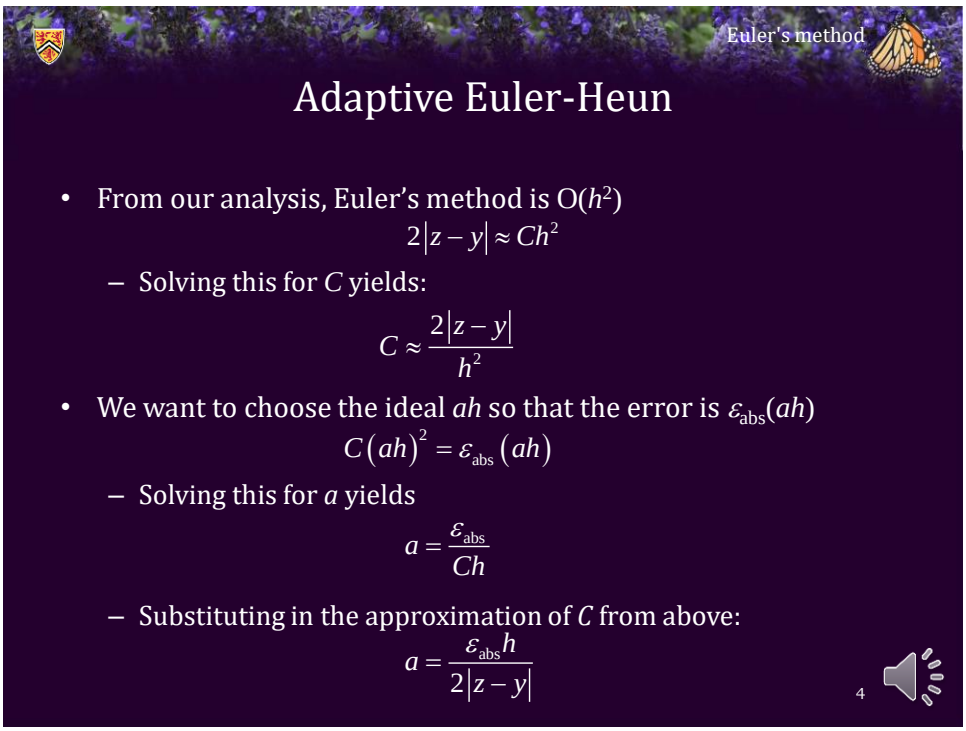
Euler's method
Adaptive Euler-Heun
• From our analysis, Euler's method is O(h^2)
$2|z - y| \approx Ch^2$
– Solving this for C yields:
$C \approx \frac{2|z - y|}{h^2}$
• We want to choose the ideal ah so that the error is $\varepsilon_{abs}(ah)$
$C(ah)^2 = \varepsilon_{abs}(ah)$
– Solving this for a yields
$a = \frac{\varepsilon_{abs}}{Ch}$
– Substituting in the approximation of C from above:
$a = \frac{\varepsilon_{abs} h}{2|z - y|}$
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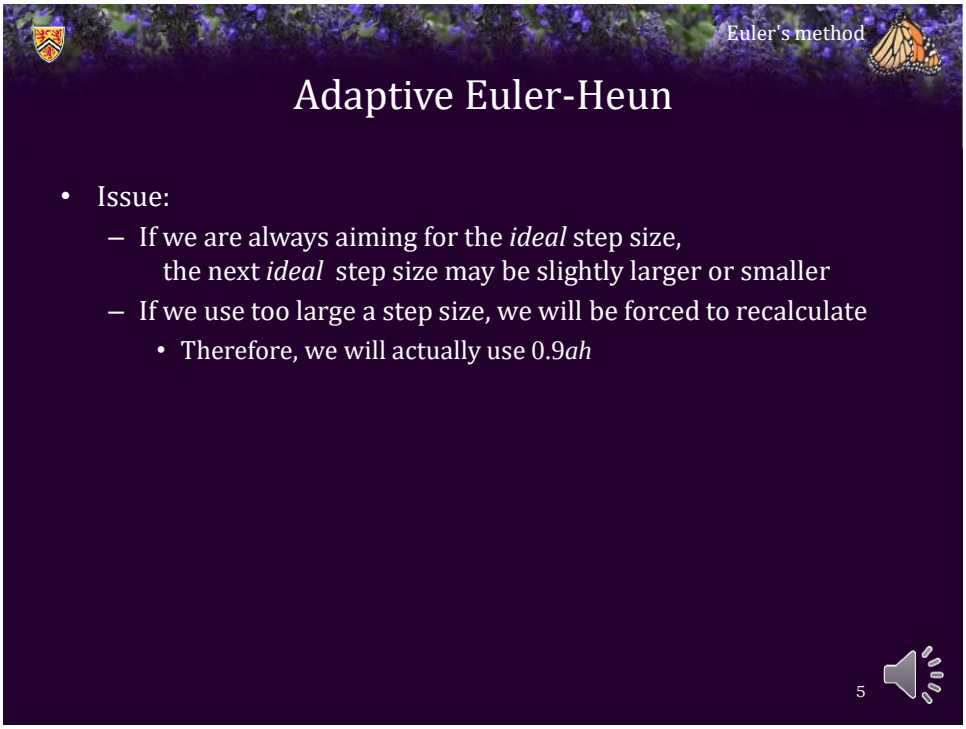
Euler's method
Adaptive Euler-Heun
• Issue:
– If we are always aiming for the *ideal* step size, the next *ideal* step size may be slightly larger or smaller
– If we use too large a step size, we will be forced to recalculate
• Therefore, we will actually use 0.9*ah*
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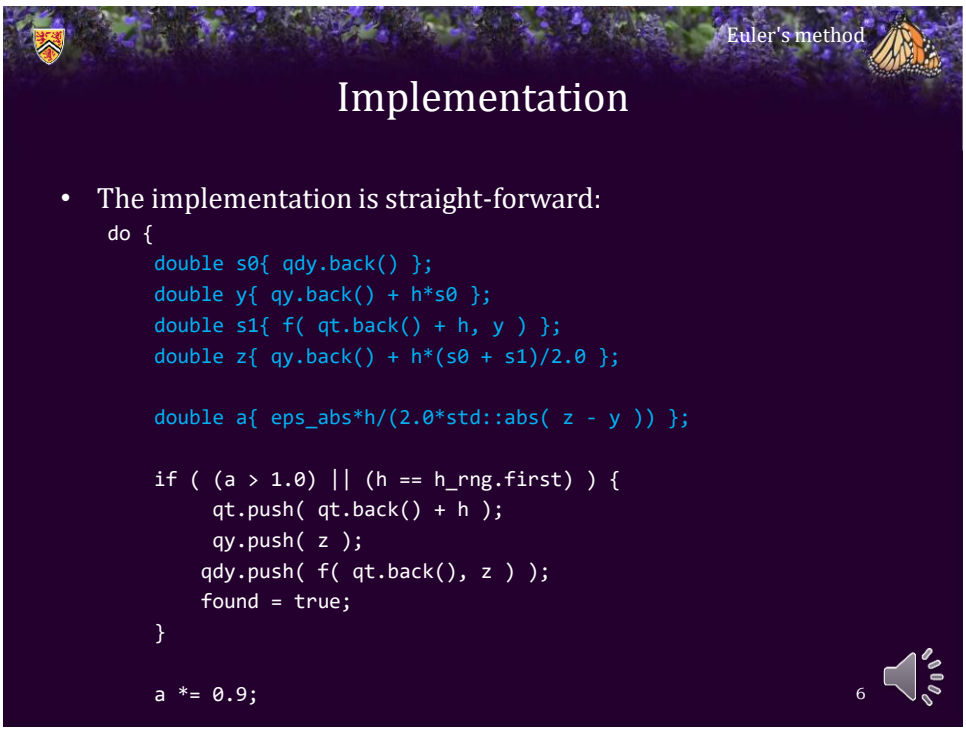
Euler's method
Implementation
• The implementation is straight-forward:
do {
double s0{ qdy.back() };
double y{ qy.back() + h*s0 };
double s1{ f(qt.back() + h, y) };
double z{ qy.back() + h*(s0 + s1)/2.0 };
double a{ eps_abs*h/(2.0*std::abs(z - y)) };
if ((a > 1.0) || (h == h_rng.first)) {
qt.push(qt.back() + h);
qy.push(z);
qdy.push(f(qt.back(), z));
found = true;
}
a *= 0.9;
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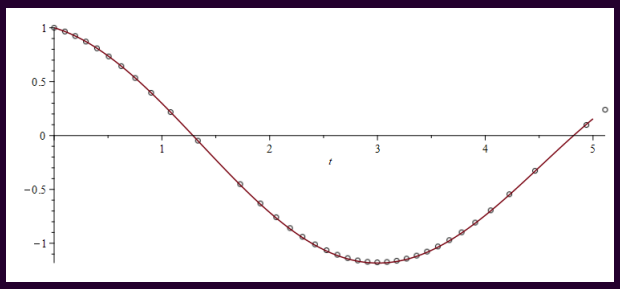
Example

- Suppose we have $y^{(1)}(t) = -0.2y(t) - \sin(t) - 0.1$

  $y(0) = 1$

  – With $h_{min} = 0.01$, $h_{max} = 1$ and $\varepsilon_{abs} = 0.1$, we have



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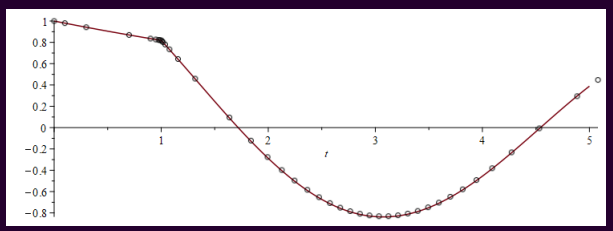
Example

- Suppose we have $y^{(1)}(t) = -0.2y(t) - \begin{cases} 0 & t < 1 \\ \sin(t) - 0.1 & t \geq 1 \end{cases}$

  $y(0) = 1$

  – With $h_{min} = 0.01$, $h_{max} = 1$ and $\varepsilon_{abs} = 0.1$, we have



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Summary

- Following this topic, you now
  - Understand the adaptive Euler-Heun method
  - Are aware of the derivation
  - Have seen the implementation
  - Have seen two examples

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References

[1] https://en.wikipedia.org/wiki/Euler_method
[2] https://en.wikipedia.org/wiki/Heun%27s_method
[3] https://en.wikipedia.org/wiki/Adaptive_algorithm
[4] https://en.wikipedia.org/wiki/Adaptive_step_size

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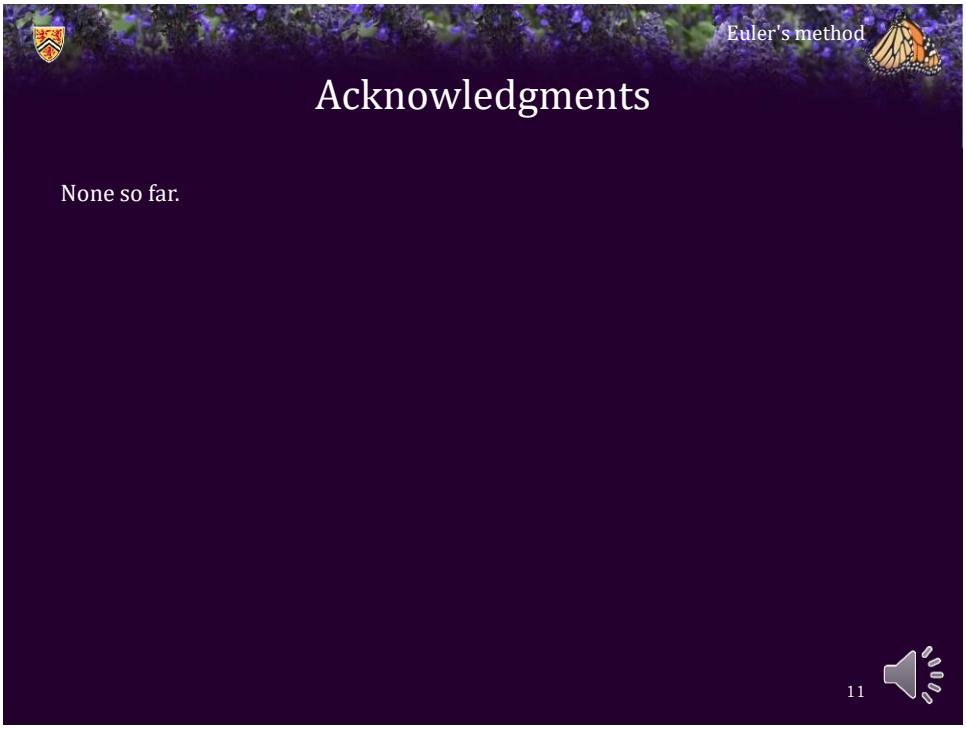
Euler's method
Acknowledgments
None so far.
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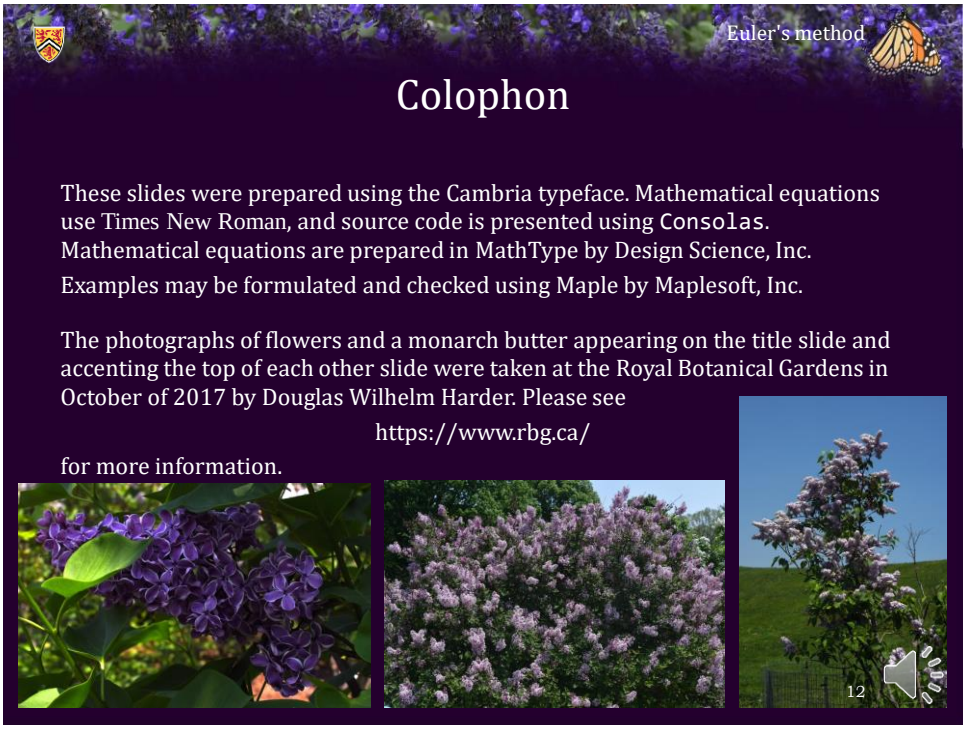
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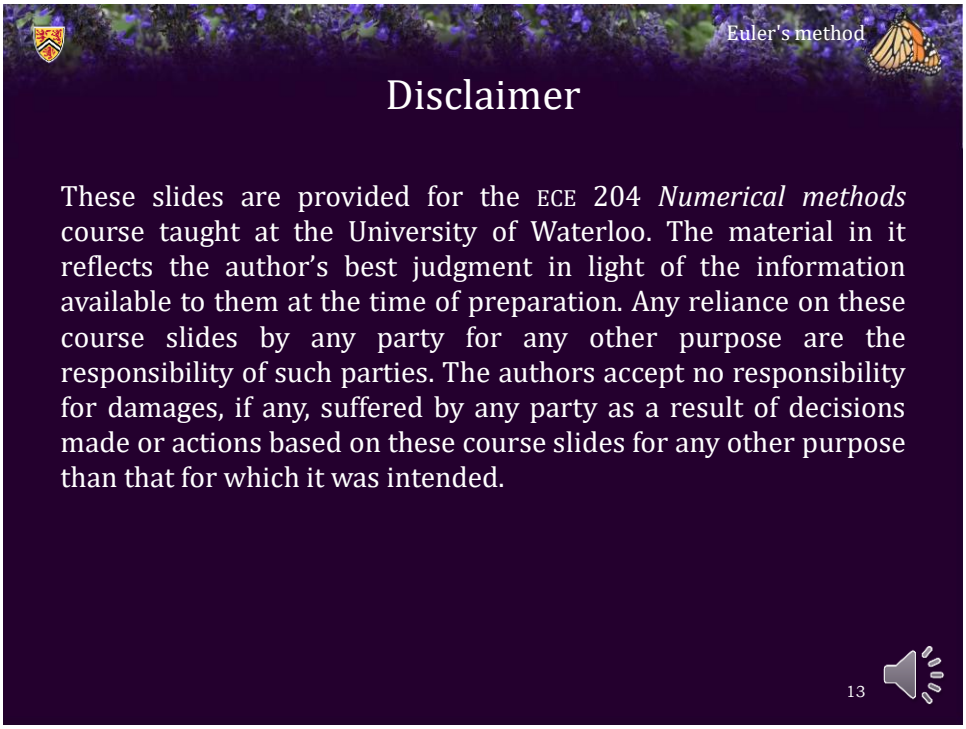
Euler's method
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